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Substitution rules for aperiodic sequences of the cut and project type

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Abstract. We consider one-dimensional aperiodic sequences arising from a cut-and-project scheme with quadratic unitary Pisot numbers β. A construction of the substitution rule is described under rather general assumptions. It allows one to build a given cut-and-project sequence $\Sigma_\beta(\Omega)$ starting from its arbitrary point. For a sequence with a convex acceptance window interval Ω, we prove that a substitution rule exists precisely if the boundary points of Ω are in the corresponding quadratic field $\mathbb{Q}[\beta]$. Typically such a substitution has a reducible characteristic polynomial. Our main result is an algorithm for construction of such a substitution rule. Some examples are shown.

1. Introduction

Ordered aperiodic structures have often been considered in physics literature in connection with modelling of the structure of physical materials (quasicrystals) and as playgrounds for certain dynamical models. One such frequently studied physical model is the one-dimensional Schrödinger equation with quasiperiodic potentials derived from Fibonacci-like sequences. (For details, refer to [15] and references therein.) Most recently, they proved useful in modifying essentially any pseudo-random number generator in order to assure its true aperiodicity [5], for another aperiodic pseudo-random number generator using related sequences see [16].

A standard way to construct quasiperiodic sequences is the cut- and-project method. In this method one projects a section of an n-dimensional periodic lattice to a suitable subspace. The sequences obtained by the cut-and-project scheme we call here simply 'quasicrystals'. They have many remarkable properties; among them, especially, is the presence of a rich structure of self-similarities [2, 9] under certain conditions on the projecting scheme and this is exploited in this paper.

Another way of generating one-dimensional quasiperiodic sequences is using the substitution rules. A substitution rule is an alphabet, together with a mapping, which assigns a finite word in the alphabet to each letter of the alphabet. A fixed point of the substitution is an infinite word w which is invariant with respect to the substitution. It can be studied from a combinatorial point of view (configurations of possible finite subwords), or from a geometrical point of view [1, 14]. In the latter case we consider the letters of w to be tiles of given length put in a prescribed order on the real line. In such a way, the tiling determines a Delone point set $\Lambda \subset \mathbb{R}$. (A set $\Lambda \subset \mathbb{R}$ is said to satisfy the Delone property if it has uniform lower and upper bounds on the distances between adjacent points.) In order that the set Λ, obtained from the substitution, is self-similar, i.e. that there exists a factor s such that $s\Lambda \subset \Lambda$, one has to choose the tiles so that they have suitable lengths. A substitution rule carries a self-similarity factor for the corresponding Delone point set as a Perron–Frobenius eigenvalue of the substitution matrix.

Some results about the connection of substitutive sequences to quasicrystals have been obtained already by Bombieri and Taylor. In [3] they show that if the characteristic polynomial of the substitution rule is the minimal polynomial of a Pisot number, then the point set given by such a substitution is a subset of a cut-and-project set. The most frequently mentioned example of the Fibonacci chain, constructed by the rule $a \to ab$, $b \to a$, is a one-dimensional cut-and-project sequence; see [8].

Gähler and Klitzing studied the Fourier transform of Delone sets generated by substitution rules. They show in [4] that a substitution determines a set with non-trivial Bragg spectrum iff the largest eigenvalue of the substitution matrix (= scaling factor of the substitution tiling) is a Pisot number.

Some authors [6, 15] have focused on studying two-symbol sequences given by the canonical projection method which can be generated using invertible substitution rules. A characteristic polynomial of invertible substitutions on two symbols is a minimal polynomial of a quadratic unitary Pisot number. A thorough description of properties of a large class of two-symbol sequences is found in [7]. In particular, the authors study generalizations of the Fibonacci chain, their Fourier transforms, relation to the cut-and-projection method, and provide an overview of their applications in physics. In addition they describe atomic surfaces of the quasiperiodic self-similar chains generated by substitutions on n symbols. In all the cases, the substitution considered has an irreducible characteristic polynomial, which means that the degree of the self-similarity factor of the generated sequence is equal to the cardinality of the alphabet of the substitution.

In this paper we consider those cut-and-project sets which arise by projection of chosen points of a two-dimensional grid $\mathbb{Z}^2$ to a straight line. In such a case the requirement of a self-similarity forces the cut-and-project sequence to be embedded into an extension of rational numbers by a quadratic irrationality. The results of this paper apply for quadratic unitary Pisot numbers, the most famous of which is the golden mean $\tau = \frac{1}{2}(1 + \sqrt{5})$. The algebraic definition of cut- and-project quasicrystals used in this paper (definition 2.3) follows the analogy of [11]. A quasicrystal is a subset of a ring of integers $\mathbb{Z}[\beta]$ of the algebraic extension $\mathbb{Q}[\beta]$ of $\mathbb{Q}$ by a quadratic Pisot unit β. A point $x \in \mathbb{Z}[\beta]$ belongs to the quasicrystal $\Sigma_\beta(\Omega)$ if its Galois conjugate x' lies in a chosen 'acceptance interval' Ω. Unlike the previously considered cases [6, 15], the acceptance window is not limited to a unit cell—our definition allows any bounded interval.

In order that a Delone set $\Lambda \subset \mathbb{R}$ with a finite number of tiles (distances between adjacent points) could be generated from a point $x \in \Lambda$ by a substitution with the eigenvalue s, Λ has to satisfy $s(\Lambda - x) \subset \Lambda - x$. (Of course, it is not a sufficient condition for the existence of a substitution rule.) A cut-and-project quasicrystal $\Sigma_\beta(\Omega)$ has the property that for any $x \in \Sigma_\beta(\Omega)$ there are infinitely many different scaling factors s, such that $s(\Sigma_\beta(\Omega) - x) \subset \Sigma_\beta(\Omega) - x$. Such points x are centres of scaling symmetry also called s-inflation centres.

In this paper we determine for each cut-and-project quasicrystal based on a quadratic unitary Pisot number β an appropriate alphabet and substitution rules whenever they exist. We prove that the substitution rules exist if and only if the end-points of the acceptance window Ω

are in the corresponding quadratic number field $\mathbb{Q}[\beta]$. It turns out that due to the abundance of self-similarities [9], if the condition on Ω is satisfied, one may find a substitution which generates the quasicrystal sequence starting from any of its points. The substitution rules depend very much on the chosen seed point $x \in \Sigma_\beta(\Omega)$ of the sequence.

It is well known that generic sequences $\Sigma_\beta(\Omega)$ have three distinct tiles, each with its own density in the infinite sequence. Varying continuously the length $|\Omega|$ of the acceptance window, the densities vary continuously. One of the densities may become zero for isolated values of $|\Omega|$—then the sequence has only two distinct tiles. Consequently, one would expect the substitution rules to be defined on three or two letters. However, during the intermediate stages of the derivation of our rules, we are led to consider much larger alphabets, distinguishing several kinds of equal-length tiles. Typically such a substitution therefore has a reducible characteristic polynomial, the Perron–Frobenius eigenvalue being a quadratic number.

Finally, let us mention that the proof of our main result (theorem 4.10) is constructive. We give the consecutive steps of the procedure for construction of the substitution rules for a given Ω and a seed point in $\Sigma_\beta(\Omega)$ in section 6.

2. Preliminaries

A substitution is a rule that assigns to each letter of an alphabet $\mathcal{A}$ a concatenation of letters, which is called a word. The set of words with letters in $\mathcal{A}$ is denoted by $\mathcal{A}^*$. Iterations of the substitution starting from a single initial letter leads to words of increasing length. Certain assumptions on the substitution rule ensure that the words give rise to an infinite sequence of letters which is invariant with respect to the substitution.

Definition 2.1. *Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be an alphabet. A map $\theta : \mathcal{A}^* \to \mathcal{A}^*$ is called a substitution, if $\theta(a)$ is non-empty for all $a \in \mathcal{A}$ and if it satisfies $\theta(vw) = \theta(v)\theta(w)$ for all $v, w \in \mathcal{A}^*$. An infinite sequence $u \in \mathcal{A}^*$ is called a fixed point of θ, if $\theta(u) = u$.*

Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be an alphabet. To a substitution $\theta : \mathcal{A}^* \to \mathcal{A}^*$ one associates a non-negative integer matrix P of order k in the following way:

$$P_{ij} := \#\text{ letters } a_j \qquad \text{in } \theta(a_i).$$

If P^n is positive for some $n \in \mathbb{N}$ (i.e. the matrix is indecomposable), the substitution is called primitive. According to the Perron–Frobenius theorem, the dominant (Perron–Frobenius) eigenvalue of such a matrix is positive and the corresponding (Perron–Frobenius) eigenvector is unique and has positive components.

We assume that there exist indices j, ℓ among $1, \dots, k$, such that $\theta(a_j) = ua_j$ and $\theta(a_\ell) = a_\ell v$ for some non-empty words $u, v \in \mathcal{A}^*$. Then by repeated application of θ on the pair $a_j|a_\ell$ of letters separated by the delimiter | we obtain words $\theta^{(n)}(a_j)|\theta^{(n)}(a_\ell)$, $n \in \mathbb{N}$, of increasing length. Clearly,

$$\theta^{(n)}(a_j)|\theta^{(n)}(a_\ell) = u_n\theta^{(n-1)}(a_j)|\theta^{(n-1)}(a_\ell)v_n$$

for certain words $u_n, v_n \in \mathcal{A}^*$. We denote by $\theta^\infty(a_j)|\theta^\infty(a_\ell)$ the bidirectional infinite word w to which $\theta^{(n)}(a_j)|\theta^{(n)}(a_\ell)$ converges. It is clear that w is invariant with respect to the substitution θ, or equivalently, w is a fixed point of θ.

To the letters of $\mathcal{A}$ one may assign lengths $\omega(a_i)$. Then it is possible to display the bidirectional infinite word w on the real line in the way that we identify the delimiter | with the origin and compose the tiles of lengths $\omega(a_i)$ on the line in the order given by the order of letters in the word w. The word w is thus a labelling of tiles of a tiling of $\mathbb{R}$. This tiling is determined by a countable point set Λ containing 0, having only a finite number of distances between neighbouring points. We further say that a countable set $\Lambda \subset \mathbb{R}$ containing 0 is generated by a substitution rule, if there exists a substitution θ on a finite alphabet $\mathcal{A}$ and a mapping $\omega\colon \mathcal{A} \to \mathbb{R}^{>0}$, such that Λ is the set obtained by displaying a fixed point w of θ on the real line using ω.

Having assigned the lengths $\omega(a_i)$ to the letters of $\mathcal{A}$, it is possible to define the length $\omega(u)$ of any finite word $u \in \mathcal{A}^*$, as a sum of lengths of its letters. The assignment of lengths may be chosen arbitrarily. We are interested in such $\omega\colon \mathcal{A}^* \to \mathbb{R}^{>0}$ that the corresponding countable set Λ is self-similar. If we choose ω such that

$$\vartheta\omega(a_i) = \omega(\theta(a_i)) \qquad \text{for} \quad i = 1, \dots, k$$

for some factor $\vartheta > 1$, then the set Λ generated by the substitution θ is self-similar, i.e. $\vartheta\Lambda \subset \Lambda$. We say that the substitution θ has the factor ϑ. Note that the factor ϑ is in fact the Perron eigenvalue of the substitution matrix of θ and the lengths $\omega(a_i)$ are components of the Perron eigenvector.

Example 2.2. As an example, let us mention the well known Fibonacci substitution rule θ on the alphabet formed by two letters, $\mathcal{A} = \{a, b\}$, given by $a \mapsto ab$, $b \mapsto a$. The second iteration $\tilde{\theta}$ of θ is the substitution

$$\begin{aligned} a &\mapsto aba \\ b &\mapsto ab \end{aligned} \tag{1}$$

therefore we can generate from the pair $a|a$ a bidirectional infinite word

$$\tilde{\theta}^\infty(a)|\tilde{\theta}^\infty(a) = \dots abaababaabaababaababa|abaababaabaababaababa \dots. \tag{2}$$

Using for a and b the lengths determined by the standard analysis of the substitution matrix, the Delone set corresponding to the Fibonacci chain has a self-similarity $\theta\Lambda \subset \Lambda$. The matrix of the substitution of (1) its Perron–Frobenius eigenvalue and the associated eigenvector are given by

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \tau^2 \qquad v = \begin{pmatrix} \tau^2 \\ \tau \end{pmatrix}$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden mean. Setting the lengths $\omega(a)$, $\omega(b)$ to be τ^2 and τ, it can be shown that the Delone set arising from the substitution coincides with a one-dimensional cut-and-project quasicrystal with a specific acceptance window. The definition is given below.

An algebraic integer β is called Pisot, if $\beta > 1$ and all its conjugates are in modulus smaller than 1. It is called unitary, or unit, if the absolute coefficient of its minimal polynomial is equal to ± 1. All quadratic unitary Pisot numbers are thus given as the larger roots of the equation

$$x^2 = mx + 1 \quad m \in \mathbb{N} \qquad \text{or} \qquad x^2 = mx - 1 \quad m \in \mathbb{N} \quad m \geqslant 3.$$

We denote by β' the algebraic conjugate of β. We have $\beta > 1 > |\beta'|$. On the field $\mathbb{Q}[\beta]$ we have the Galois automorphism $'$ defined by $x' = (a + b\beta)' := a + b\beta'$, for $a, b \in \mathbb{Q}$. We shall consider the ring $\mathbb{Z}[\beta] := \mathbb{Z} + \mathbb{Z}\beta$, which is a (possibly trivial) subring of the ring of integers in $\mathbb{Q}[\beta]$.

Definition 2.3. *Let β be a quadratic unitary Pisot number and $\Omega \subset \mathbb{R}$ a bounded interval with non-empty interior. A cut-and-project quasicrystal is the set*

$$\Sigma_\beta(\Omega) := \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\}$$

and Ω is called the acceptance window (interval) of the quasicrystal $\Sigma_\beta(\Omega)$. We sometimes omit the subscript β if there is no danger of confusion.

Let us formulate some of the properties of quasicrystals which will be used in this paper. Their proof is a direct consequence of the definition. Note that for (v) of lemma 2.4 it is essential that β is a unit and therefore $\pm\beta^k$, $k \in \mathbb{Z}$, are divisors of unity in the ring $\mathbb{Z}[\beta]$, which ensures that $\mathbb{Z}[\beta] = \beta^k\mathbb{Z}[\beta]$.

Lemma 2.4. *Let β be a quadratic unitary Pisot number and let Ω_1, Ω_2, and $\Omega \subset \mathbb{R}$ be intervals. Then*

(i) $\Sigma_\beta(\Omega_1) \subset \Sigma_\beta(\Omega_2)$ *if* $\Omega_1 \subset \Omega_2$*;*
(ii) $\Sigma_\beta(\Omega_1) \cap \Sigma_\beta(\Omega_2) = \Sigma_\beta(\Omega_1 \cap \Omega_2)$*;*
(iii) $\Sigma_\beta(\Omega_1) \cup \Sigma_\beta(\Omega_2) = \Sigma_\beta(\Omega_1 \cup \Omega_2)$*;*
(iv) $\Sigma_\beta(\Omega) + \eta = \Sigma_\beta(\Omega + \eta')$*, for any* $\eta \in \mathbb{Z}[\beta]$*;*
(v) $\beta^k\Sigma_\beta(\Omega) = \Sigma_\beta(\beta'^k\Omega)$*, for* $k \in \mathbb{Z}$*.*

Consequently, we shall be interested only in quasicrystals with acceptance windows $\Omega = [c, d)$. The restriction that the interval is semiclosed, $[c, d)$, may influence only presence or absence of two particular points, namely c' and d' in the quasicrystal. This remark applies, of course, only if c, d belong to $\mathbb{Z}[\beta]$. Otherwise, including the boundary of the acceptance interval is not relevant for the quasicrystal. It is obvious that adding or removing a single point to a point set may strongly influence the form of the substitution rule. We deal with this problem in section 5, providing examples of substitution rules for quasicrystals with closed and open acceptance windows.

Construction of a substitution rule for a given quasicrystal $\Sigma_\beta(\Omega)$ from its particular point $x \in \Sigma_\beta(\Omega)$ is, according to (iv) of lemma 2.4, a task equivalent to finding a rule for $\Sigma_\beta(\Omega - x')$ starting from zero. Therefore we can limit ourselves to quasicrystals containing the origin, which gives us the conditions $c \leqslant 0$, $d > 0$, on the acceptance interval $\Omega = [c, d)$.

A further important property of quasicrystals, which is essential for this paper, is the fact that there are only finitely many distances between adjacent points, or in other words, finitely many distinct tiles. The distances for a particular acceptance window may be determined using the following proposition, taken from [10].

Proposition 2.5. *Let β be a quadratic unitary Pisot number. The distances between adjacent points of the model set $\Sigma_\beta[c, d)$ take only two or three values, according to the length $d - c$ of the acceptance interval. We put*

$$\phi_j(\beta) := \begin{cases} \beta - j & \text{for} \quad j = 0, 1, \ldots, [\beta] - 1 \\ 1 & \text{for} \quad j = [\beta] \end{cases}$$

and

$$L_{j,\beta} = j - \beta' \qquad \text{for} \quad j = 1, \ldots, [\beta].$$

Then

for $d - c \in (\phi_j(\beta), \phi_{j-1}(\beta))$, $j = 1, \ldots, [\beta]$, *the distances are* 1, $L_{j,\beta}$ *and* $L_{j,\beta} + 1$,
for $d - c = \phi_{j-1}(\beta)$, $j = 1, \ldots, [\beta]$, *the distances are two* 1, $L_{j,\beta}$.

The above proposition describes the tiles in $\Sigma_\beta[c, d)$ with $d - c \in (1, \beta]$ only. Note that for other lengths of the acceptance window one may derive the results easily using (v) of lemma 2.4. For the purposes of this paper we shall not need the exact values of the distances and thus the complicated notation introduced in the proposition. Our results are based on the following lemma, which is a direct consequence of the stated facts.

Lemma 2.6. *Let β be a quadratic unitary Pisot number, $c, d \in \mathbb{R}$. We index the points of $\Sigma_\beta[c, d)$, such that $\Sigma_\beta[c, d) = \{x_n \mid n \in \mathbb{Z}\}$, $x_n < x_{n+1}$. Then $x_{n+1} - x_n$ take at most three values, say $L, M, S \in \mathbb{Z}[\beta]$, standing for long, middle and short tile, respectively. (We consider $L > M > S$.) We have $S'M' < 0$, i.e. S' and M' have opposite signs. We denote $\Sigma_X := \{x_n \mid x_{n+1} - x_n = X\}$ for $X = S, M, L$. Then: $\Sigma[c, d) = \Sigma_S \cup \Sigma_M \cup \Sigma_L$, where*

$$\begin{aligned} \Sigma_S &= \{x \in \mathbb{Z}[\beta] \mid x' \in [c, d - S')\} \\ \Sigma_M &= \{x \in \mathbb{Z}[\beta] \mid x' \in [c - M', d)\} \qquad \text{if} \quad S' > 0 \\ \Sigma_L &= \{x \in \mathbb{Z}[\beta] \mid x' \in [d - S', c - M')\} \end{aligned}$$

or

$$\begin{aligned} \Sigma_S &= \{x \in \mathbb{Z}[\beta] \mid x' \in [c, d - M')\} \\ \Sigma_M &= \{x \in \mathbb{Z}[\beta] \mid x' \in [c - S', d)\} \qquad \text{if} \quad S' < 0 \\ \Sigma_L &= \{x \in \mathbb{Z}[\beta] \mid x' \in [d - M', c - S')\}. \end{aligned}$$

Let us point out that Σ_X contains those points of the quasicrystal which are left end-points of the tile with length X. Note that $\Sigma_X = \Sigma_\beta(\Omega_X)$ is a quasicrystal according to definition 2.3, if the corresponding window Ω_X is non-empty. However, according to proposition 2.5, for certain lengths $d - c$ of the interval $[c, d)$, the distances between adjacent points in the quasicrystal $\Sigma_\beta[c, d)$ take only two values, say S and M, and thus the set Σ_L vanishes.

3. Construction of substitutions—idea and examples

In this section we explain the main ideas of the paper with regard to construction of a substitution for a given quasicrystal. We illustrate it on two examples. In the first of these (example 3.1) we take the well known Fibonacci substitution and explain the fact that the corresponding Delone set is a cut-and-project quasicrystal. In the second example 3.2, we start with a chosen quasicrystal and demonstrate the method of how to find a substitution rule for it.

A quasicrystal with tiles S, M, L is a bidirectional infinite word in these letters. Fixing a point in a quasicrystal corresponds to fixing the delimiter | between two particular letters of the word. Construction of a substitution for such a quasicrystal with respect to a chosen point stems in finding a rule θ, such that the word of the quasicrystal, with a given delimiter |, is its fixed point. In general, only for a certain family of quasicrystals we find a substitution rule having the alphabet with three letters $\{S, M, L\}$. For other quasicrystals, the tiles can be divided into groups $S_1, \ldots, S_{k_S}$, $M_1, \ldots, M_{k_M}$, and $L_1, \ldots, L_{k_L}$. Then there exists a substitution θ with alphabet $\mathcal{A} = \{S_1, \ldots, L_{k_L}\}$ such that the quasicrystal word is its fixed point. More formally, it is possible to find a substitution θ with an alphabet $\mathcal{A}$ and an assignment of lengths $\omega : \mathcal{A} \to \{S, M, L\}$ such that the quasicrystal is the Delone set corresponding to some fixed point of θ.

We shall construct the substitution rule by dividing the short, middle and long distances into several groups in such a way that tiles in one group are, after rescaling $\Sigma_\beta(\Omega)$ by a suitable scaling factor, filled by the same sequence of distances. This is illustrated in figure 1.

The determination of left end-points of tiles S, M, or L consists, according to lemma 2.6, of splitting the acceptance window of the quasicrystal into three disjoint subintervals. Similarly, determination of left end-points of $S_1, \ldots, S_{k_S}$, $M_1, \ldots, M_{k_M}$, or $L_1, \ldots, L_{k_L}$ corresponds to a splitting of the acceptance window Ω into $k_S + k_M + k_L$ disjoint intervals $\Omega_1, \ldots, \Omega_{k_L}$. We show that two quasicrystal points which have their Galois image in the same subinterval Ω_i are, after rescaling by the scaling factor, left-end points of the same sequence of distances filling the tile S, M, or L. We call the points determining the division of Ω into Ω_i the splitting points.

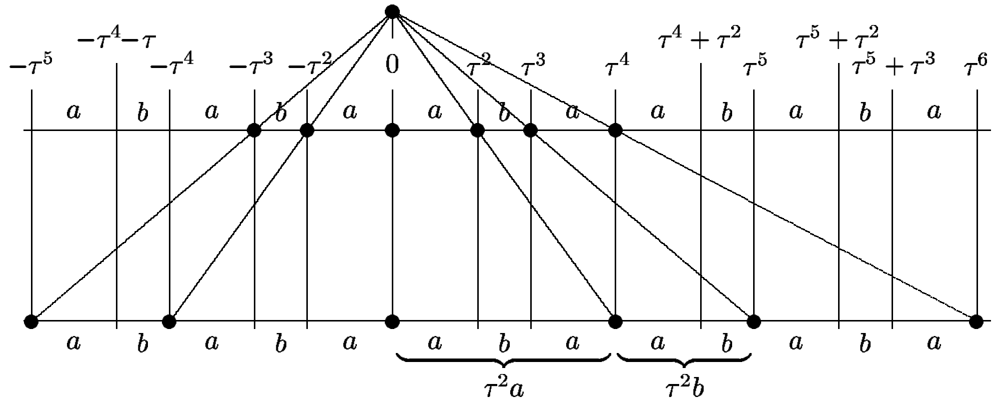


Figure 1. The quasicrystal $\Sigma[-1/\tau^2, 1/\tau)$ is generated by the Fibonacci substitution rule: we stretch the quasicrystal by a suitable scaling factor (τ^2) and compose the stretched tiles by tiles of the original quasicrystal. Stretched tiles of the same length should be composed of the same sequence of original tiles.

We provide an explicit prescription for finding the splitting. It is obvious that if one has a substitution for a given quasicrystal and its point, then it is possible to find another substitution with larger alphabet for the same scaling factor. For example, instead of a letter a we may write a_1, a_2 in a random order. However, the division of Ω using splitting points provides a substitution rule with minimal alphabet for the given scaling factor.

It can be recognized from the form of the acceptance window Ω whether the set of splitting points is finite or infinite; see proposition 4.9. If it is infinite, then for a given quasicrystal a substitution rule with the given factor does not exist. We then show that one cannot find a substitution with another scaling factor, (theorem 4.10).

If the set of splitting points is finite, it determines the number of letters of the alphabet. The size of the alphabet might be reduced using another scaling factor, for example by iteration of the given substitution. This prolongs the word assigned to a single letter by the substitution. At the end of section 4 we illustrate such a possibility on a specific example.

Example 3.1. We recall the Fibonacci substitution rule (1). We recall that the dominant eigenvalue of the substitution matrix is τ^2, where τ is the golden mean. We show the relation of the Fibonacci chain to the quasicrystal $\Sigma_\tau[-1/\tau^2, 1/\tau)$.

Now let $\beta = \tau$. Using the proposition 2.5 we may determine distances (length of tiles) in the quasicrystal. The length of the acceptance window is $\tau^{-2} + \tau^{-1} = 1$, thus we have to use (v) of lemma 2.4 to see that $\Sigma[-1/\tau^2, 1/\tau) = -\tau\Sigma[-1/\tau, 1)$. The length of the new window is $1 + \frac{1}{\tau} = \tau = \phi_0(\tau)$. We have $[\tau] = 1$, and thus there are two distances in $\Sigma[-1/\tau, 1)$: 1 and $L_{1,\tau} = \tau$. Distances in $\Sigma[-1/\tau^2, 1/\tau)$ are just τ multiples of 1 and τ, i.e. τ and τ^2.

Consider the letters a, b in the fixed point (2) to be tiles dividing the real axis. We set the length of the tile represented by a to be τ^2, and the length of the tile b to τ. Let us show that the tiling sequence $\tilde\theta^\infty(a)|\tilde\theta^\infty(a)$ produces the quasicrystal $\Sigma[-1/\tau^2, 1/\tau)$. The procedure is illustrated in figure 1.

We shall use again the self-similarity property (v) of lemma 2.4:

$$\tau^2\Sigma\left[-\frac{1}{\tau^2}, \frac{1}{\tau}\right) = \Sigma\left[-\frac{1}{\tau^4}, \frac{1}{\tau^3}\right) \subset \Sigma\left[-\frac{1}{\tau^2}, \frac{1}{\tau}\right).$$

The last inclusion is valid due to (i) of the lemma. Altogether this means that the quasicrystal point set rescaled with respect to 0 by the scaling factor τ^2 is a subset of the original quasicrystal. The quasicrystal $\Sigma[-1/\tau^4, 1/\tau^3)$ has the same ordering of tiles as $\Sigma[-1/\tau^2, 1/\tau)$, but the lengths are τ^2 times rescaled, i.e. τ^4 and τ^3. The points of $\Sigma[-1/\tau^2, 1/\tau)$ divide the tiles of $\Sigma[-1/\tau^4, 1/\tau^3)$. Clearly, the rescaled tiles (lengths τ^4, τ^3) are divided into tiles of lengths τ^2 and τ. It is possible to do this only in such a way that $\tau^4 = 2\tau^2 + \tau$ is cut into two lengths of τ^2 and one length of τ; similarly the tile of length τ^3 splits into one length of τ^2, and one length of τ (recall that $\tau^3 = \tau^2 + \tau$). Therefore the longer tile a after rescaling is replaced by the concatenation of twice a and b, the rescaled b becomes a concatenation of a and b. We will show that each of letters a are replaced by aba in the same order; similarly every b is replaced by ab. We will thus show that $\Sigma[-1/\tau^2, 1/\tau)$ is generated by the substitution $\tilde{\theta}(a) = aba$, $\tilde{\theta}(b) = ab$.

Note that the substitution $\tilde{\theta}$ is given by the action of several affine mappings. First, we rescale the points by the factor τ^2. This corresponds to the mapping $t_{(1)}x := \tau^2 x$. Then we split the enlarged tiles by adding new points. Let x be the left end-point of a tile a. The splitting of the enlarged tile with left end-point being $\tau^2 x$ according to $\tilde{\theta}(a) = aba$ means inserting new successors to the point $\tau^2 x$, by mappings

$$t_{(2)}x := \tau^2 x + \tau^2$$
$$t_{(3)}x := \tau^2 x + \tau^2 + \tau.$$

Now let y be the left end-point of a tile b. The splitting of the rescaled tile b with the left end-point being $t_{(1)}y = \tau^2 y$ according to $\tilde{\theta}(b) = ab$ is done by adding one new point $t_{(2)}y := \tau^2 y + \tau^2$.

Now it is important to decide which quasicrystal points are left end-points of tiles a with length τ^2, and which of them are left end-points of tiles of the type b with length τ. Using lemma 2.6 for the given quasicrystal $\Sigma[-1/\tau^2, 1/\tau)$ we find that $y \in \Sigma_M = \Sigma[1/\tau^3, 1/\tau)$ are left end-points of tiles b and $x \in \Sigma_L = \Sigma[-1/\tau^2, 1/\tau^3)$ are the left end-points of tiles a. Note that $\Sigma_S = \emptyset$ in this case.

In order to show that the substitution $\tilde{\theta}$ generates our quasicrystal we have to prove that

$$\Sigma\left[-\frac{1}{\tau^2}, \frac{1}{\tau}\right) = t_{(1)}\Sigma_M \cup t_{(2)}\Sigma_M \cup t_{(1)}\Sigma_L \cup t_{(2)}\Sigma_L \cup t_{(3)}\Sigma_L. \tag{3}$$

Applying (iv) and (v) of lemma 2.4, we obtain

$$t_{(1)}\Sigma_M = \tau^2\Sigma_M = \Sigma\left[\frac{1}{\tau^5}, \frac{1}{\tau^3}\right)$$
$$t_{(1)}\Sigma_L = \tau^2\Sigma_L = \Sigma\left[-\frac{1}{\tau^4}, \frac{1}{\tau^5}\right)$$
$$t_{(2)}\Sigma_M = \tau^2\Sigma_M + \tau^2 = \Sigma\left[\frac{1}{\tau^5} + \frac{1}{\tau^2}, \frac{1}{\tau}\right)$$
$$t_{(2)}\Sigma_L = \tau^2\Sigma_L + \tau^2 = \Sigma\left[\frac{1}{\tau^3}, \frac{1}{\tau^5} + \frac{1}{\tau^2}\right)$$
$$t_{(3)}\Sigma_L = \tau^2\Sigma_L + \tau^3 = \Sigma\left[-\frac{1}{\tau^2}, -\frac{1}{\tau^4}\right).$$

The statement (iii) of proposition 2.4 then gives us the result (3).

In the previous example we have shown how the Fibonacci substitution (1) generates the tile quasicrystal $\Sigma_\tau[-1/\tau^2, 1/\tau)$, which has two different distances between adjacent points. In the following example we find a substitution rule for the quasicrystal $\Sigma_\tau[0, 1 + 1/\tau^2)$, where the distances between adjacent points take three different values. Although the quasicrystal has only three types of tiles, we will use an alphabet formed by four letters. It will be clear that with the chosen scaling factor τ^2, one cannot find a substitution rule with a smaller alphabet.

Example 3.2. Consider the quadratic Pisot unit τ and the quasicrystal $\Sigma[0, 1+1/\tau^2)$ in $\mathbb{Z}[\tau]$. Using proposition 2.5 we find that the distances between adjacent points in this quasicrystal are $S = 1$, $M = \tau$ and $L = \tau^2$. We recall the sets Σ_S, Σ_L and Σ_M of left end-points of tiles S, M, L, respectively. Lemma 2.6 allows us to find the bidirectional word in the alphabet $\{S, M, L\}$ corresponding to $\Sigma(\Omega)$:

$$\ldots SMLMSMMSMLMSMM|SMLMSMLMSMMSMLMSM\ldots \tag{4}$$

where the delimiter | marks the origin. Let us split the acceptance window $\Omega = [0, 1+1/\tau^2)$ into intervals Ω_i, $i = 1, \ldots, 4$ as follows:

$$\Omega_1 = [0, 1/\tau^2) \qquad \Omega_2 = [1/\tau^2, 1/\tau) \qquad \Omega_3 = [1/\tau, 1) \qquad \Omega_4 = [1, 1+1/\tau^2).$$

We have $\Sigma_S = \Sigma(\Omega_1)$, $\Sigma_L = \Sigma(\Omega_2)$, $\Sigma_M = \Sigma(\Omega_3) \cup \Sigma(\Omega_4)$. Let us assign the letters to intervals Ω_i, S to Ω_1, L to Ω_2, M_1 to Ω_3, and M_2 to Ω_4. In the construction of the substitution for our quasicrystal, we proceed in the following way: we stretch the quasicrystal by the factor τ^2 with respect to the origin. We obtain the quasicrystal $\tau^2\Sigma[0, 1+1/\tau^2) = \Sigma[0, 1/\tau^2+1/\tau^4) \subset \Sigma[0, 1+1/\tau^2)$. The tiles of the stretched quasicrystal have lengths τ^2 times greater than S, M, L, namely τ^2, τ^3 and τ^4.

Let us take a left end-point y^I of a stretched tile with length $\tau^2 S = \tau^2$. We shall look for successors $y^{II}, y^{III}, \ldots$ of y^I in $\Sigma[0, 1+1/\tau^2)$, which fill the corresponding tile $\tau^2 S$. We have $y^I \in \tau^2\Sigma_S$. Therefore the $'$ image $x = (y^I)'$ belongs to $1/\tau^2\Omega_1$. Since $1/\tau^2\Omega_1 \subset \Omega_1$, the point y^I belongs to Σ_S, and thus its first successor in $\Sigma(\Omega)$ is the point $y^{II} = y^I + S$. Therefore $(y^{II})' \in 1/\tau^2\Omega_1 + 1 \subset \Omega_4$. This means that $y^{II} \in \Sigma_M$. Its successor is $y^{III} = y^{II} + M_2$. We have $(y^{III})' \in 1/\tau^2\Omega_1 + 1 - 1/\tau \subset 1/\tau^2\Omega$. Thus y^{III} is the right end-point of the stretched tile $\tau^2 S$. Symbolically it can be written as

$$S : \left[0, \frac{1}{\tau^4}\right) \xrightarrow{S} \left[1, 1+\frac{1}{\tau^4}\right) \xrightarrow{M_2} \left[\frac{1}{\tau^2}, \frac{1}{\tau^2}+\frac{1}{\tau^4}\right) \subset \frac{1}{\tau^2}\Omega.$$

We see that a stretched tile $\tau^2 S$ is filled by tiles S and M_2, respectively. Formally, we have the rule $S \to SM_2$.

Similarly we can proceed for left end-points of other stretched tiles. It suffices to realize that the $'$ images of the left end-points of stretched tiles $\tau^2 L$ belong to $1/\tau^2\Omega_2 = [1/\tau^4, 1/\tau^3)$. For left end-points of $\tau^2 M_1$ we have $1/\tau^2\Omega_3 = [1/\tau^3, 1/\tau^2)$, and for $\tau^2 M_2$ we have $1/\tau^2\Omega_4 = [1/\tau^2, 1/\tau^2+1/\tau^4)$. The filling of the stretched tiles is schematically described by the following formulae and illustrated in figure 2:

$$\begin{aligned}
L : \left[\frac{1}{\tau^4}, \frac{1}{\tau^3}\right) &\xrightarrow{S} \left[1+\frac{1}{\tau^4}, 1+\frac{1}{\tau^3}\right) \xrightarrow{M_2} \left[\frac{1}{\tau^2}+\frac{1}{\tau^4}, \frac{1}{\tau}\right) \\
&\xrightarrow{L} \left[\frac{2}{\tau^2}+\frac{1}{\tau^4}, 1\right) \xrightarrow{M_1} \left[\frac{1}{\tau^3}+\frac{1}{\tau^6}, \frac{1}{\tau^2}\right) \subset \frac{1}{\tau^2}\Omega \\
M_1 : \left[\frac{1}{\tau^3}, \frac{1}{\tau^2}\right) &\xrightarrow{S} \left[1+\frac{1}{\tau^3}, 1+\frac{1}{\tau^2}\right) \xrightarrow{M_2} \left[\frac{1}{\tau}, \frac{2}{\tau^2}\right) \xrightarrow{M_1} \left[0, \frac{1}{\tau^4}\right) \subset \frac{1}{\tau^2}\Omega \\
M_2 : \left[\frac{1}{\tau^2}, \frac{1}{\tau^2}+\frac{1}{\tau^4}\right) &\xrightarrow{L} \left[\frac{2}{\tau^2}, \frac{2}{\tau^2}+\frac{1}{\tau^4}\right) \xrightarrow{M_1} \left[\frac{1}{\tau^4}, \frac{1}{\tau^3}+\frac{1}{\tau^6}\right) \subset \frac{1}{\tau^2}\Omega.
\end{aligned}$$

Let us now make several comments on the table in figure 2. We can see that the first and the last iterations of a given letter X, ($X \in \{S, L, M_1, M_2\}$), always belong to $1/\tau^2\Omega$. The second to last iterations X^{II}, $X^{III}, \ldots$ of all letters together cover disjointly the entire acceptance

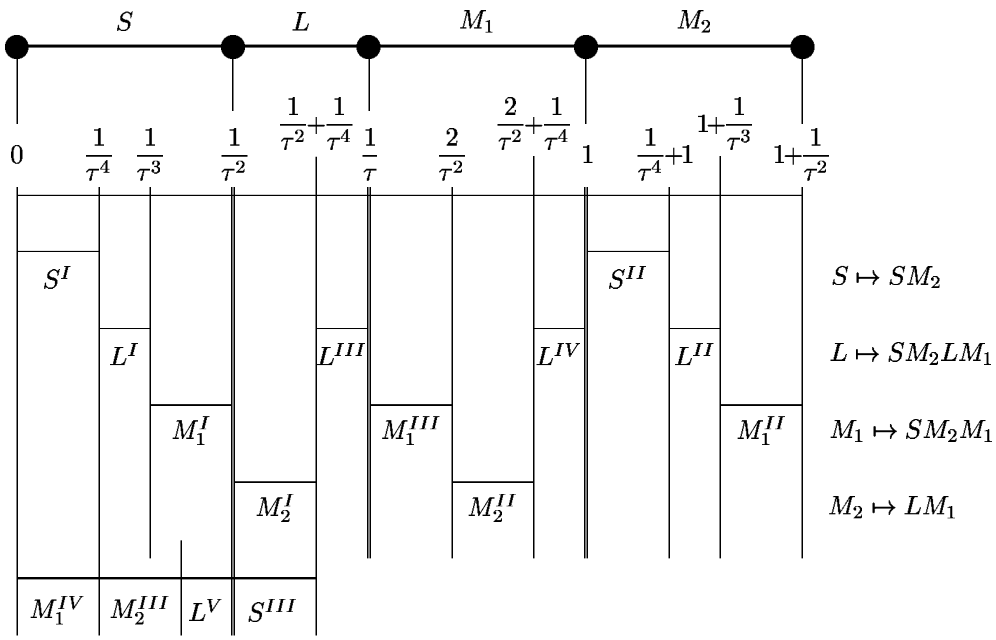


Figure 2. Schematic representation of the construction of a substitution rule for the quasicrystal $\Sigma[0, 1 + 1/\tau^2)$.

window Ω. Note that the first iteration X^I divides the interval $1/\tau^2\Omega$ in a different way to the division by the last iteration of letters. The points of X^I correspond to left end-points of stretched tiles $\tau^2 X$, and the points of the last iterations correspond to right end-points of the stretched tiles $\tau^2 X$. Note that it was necessary to split M into M_1 and M_2, since points from M_1^I and M_2^I jump according to different prescriptions.

Finally, note that in the substitution rules the prescription for S starts with S and the prescription for M_1 ends with M_1. Thus starting with the pair $M_1|S$ and iterating the substitution leads to a bidirectional infinite word which is a fixed point of the substitution

$$\ldots SM_2LM_1SM_2M_1SM_2LM_1SM_2M_1|SM_2LM_1SM_2LM_1SM_2M_1SM_2LM_1SM_2\ldots.$$

One may verify that after erasing the indices of letters M, the above word coincides with that of (4).

4. Justification of the method

In the previous section, we have explained using examples the steps which have to be done in order to construct a substitution rule for a given quasicrystal in $\mathbb{Z}[\tau]$. In this section, we consider any quadratic Pisot unit β. We provide an algorithm for finding a suitable splitting of the acceptance window to groups of points which will possess the same letter. We shall justify the procedure of construction of the substitution rules from such splittings, illustrated in example 3.2.

Throughout this section, we shall consider the acceptance window $\Omega = [c, d)$, where $c \leqslant 0$, $d > 0$. Using lemma 2.6 above, we are able to give a prescription of how to generate, one after another, the points of a quasicrystal. In each step we go from a point $y \in \Sigma_\beta[c, d)$ to its successor in $\Sigma_\beta[c, d)$. Considering the same procedure for $'$ images of points $x = y'$ in the acceptance window, we may define a function $f\colon \Omega \to \Omega$ by

$$
\begin{aligned}
f(x) &:= \begin{cases} x + S' & \text{for} \quad x \in \Omega_S := [c, d - S') \\ x + M' & \text{for} \quad x \in \Omega_L := [c - M', d) \\ x + L' & \text{for} \quad x \in \Omega_M := [d - S', c - M') \end{cases} \qquad \text{for} \quad S' > 0 \\
f(x) &:= \begin{cases} x + S' & \text{for} \quad x \in \Omega_S := [c, d - M') \\ x + M' & \text{for} \quad x \in \Omega_L := [c - S', d) \\ x + L' & \text{for} \quad x \in \Omega_M := [d - M', c - S') \end{cases} \qquad \text{for} \quad S' < 0.
\end{aligned}
\tag{5}
$$

We recall that for certain lengths $d - c$ of the acceptance window, the interval Ω_L vanishes. In this case the two discontinuity points of the function f coincide.

Note that if $x \in \mathbb{Z}[\beta]$, then $f(x) \in \mathbb{Z}[\beta]$. The function f corresponds to walking through the quasicrystal. The first successor of a quasicrystal point y is $(f(y'))'$. Generally, the kth successor of y is $(f^{(k)}(y'))'$. Note that f is in fact defined not only for numbers in $\mathbb{Z}[\beta]$, but for all $x \in [c, d)$. For $x \notin \mathbb{Z}[\beta]$, the image $f(x)$ does not have an interpretation using quasicrystal points. However, if $x \in \mathbb{Q}[\beta]$, i.e. $x \in \frac{1}{q}\mathbb{Z}[\beta]$, for some $q \in \mathbb{N}$, then also $f(x) \in \frac{1}{q}\mathbb{Z}[\beta]$. The image of $f(x)$ under the Galois automorphism is well defined and we may write

$$x' + S \leqslant (f(x))' \leqslant x' + L. \tag{6}$$

For determination of the substitution rule, we shall work with the given quasicrystal $\Sigma_\beta(\Omega)$, and with its s times rescaled copy $s\Sigma_\beta(\Omega)$. If s is the self-similarity factor of $\Sigma_\beta(\Omega)$, we have $s\Sigma_\beta(\Omega) \subset \Sigma_\beta(\Omega)$. It can be, for example, any $s \in \mathbb{Z}[\beta]$, with $0 < s' < 1$. In the following, we shall work with a factor s, which is a unit in $\mathbb{Z}[\beta]$. The minimum value of such factors is $s = \beta^2$ for $\beta^2 = m\beta + 1$ or $s = \beta$, if $\beta^2 = m\beta - 1$. The advantage of s being a unit is that $s\Sigma_\beta(\Omega) = \Sigma_\beta(s'\Omega)$. The length of tiles in $\Sigma_\beta(\Omega)$ (distances between its adjacent points) are S, M and L. The tiles in $s\Sigma_\beta(\Omega)$ have lengths sS, sM, sL. The number of points in $\Sigma_\beta(\Omega)$ between two neighbours in $s\Sigma_\beta(\Omega)$ is thus bounded by a constant, say K.

We can formulate previous observations using the function f into the following propositions. The assertions are transparent only for $x, y \in \mathbb{Z}[\beta]$. The proof for $x, y \notin \mathbb{Z}[\beta]$ follows from the fact that $\mathbb{Z}[\beta] \cap \Omega$ is dense in Ω, and f is right continuous on Ω.

Proposition 4.1. *Let $s \in \mathbb{Z}[\beta]$, $0 < s' < 1$. There exists $K \in \mathbb{N}$ such that for any $x \in s'\Omega$ there exists a positive integer $n(x) \leqslant K$, satisfying*

$$f^{(n(x))}(x) \in s'\Omega \qquad \text{and} \qquad f^{(j)}(x) \notin s'\Omega \qquad \text{for} \quad 1 \leqslant j \leqslant n(x) - 1.$$

Proposition 4.2. *Let $s \in \mathbb{Z}[\beta]$, $0 < s' < 1$, be a unit. For any $y \in \Omega$, there exists a unique $x \in s'\Omega$, and a unique $0 \leqslant p < n(x)$ such that $f^{(p)}(x) = y$.*

The function f is a bijection on Ω, therefore the inverse $f^{(-1)}$ is well defined. Thus for a given $y \in \Omega$, the x from proposition 4.2 can be found by iteration of $f^{(-1)}$. Let us define a function $g\colon \Omega \to \Omega$ by the prescription

$$g(y) := \frac{1}{s'} f^{(-p)}(y) \tag{7}$$

where p is the minimal non-negative integer j such that $f^{(-j)}(y) \in s'\Omega$. Note that p and $x := f^{(-p)}(y)$ are the ones from proposition 4.2. Thus using proposition 4.1 we have $0 \leqslant p \leqslant K - 1$. The function g will be essential further on.

Now we assume that we have a substitution rule with the scaling factor $s \in \mathbb{Z}[\beta]$, whose fixed point is the bidirectional infinite word corresponding to the given quasicrystal $\Sigma_\beta(\Omega)$. The prescription of the substitution corresponds to filling of the tiles sS, sL, sM of the stretched quasicrystal $s\Sigma_\beta(\Omega)$, by the tiles of the original $\Sigma_\beta(\Omega)$, according to the function f. We consider two elements $x, y \in \Sigma_\beta(\Omega)$, left end-points of tiles which are assigned with the same letter, say a. It means that in an arbitrary iteration, say rth, of the substitution, the s^r times stretched tiles, with left end-points $s^r x$ and $s^r y$, are filled by the same sequence of distances. This implies that the iterations of the function f behave in the same way on points $(s^r x)'$, $(s^r y)'$. It means that for any $z \in \Sigma_\beta(\Omega)$, such that $x' < z' < y'$, the function f behaves in the same way, and therefore we can assign the tile with left end-point z also by the letter a. Without loss of generality, we may assume that the points assigned to the same letter form an interval in $\Omega \cap \mathbb{Z}[\tau]$. The acceptance window Ω is split into a finite disjoint union of intervals, corresponding to letters of the alphabet.

Definition 4.3. *Let $s \in \mathbb{Z}[\beta]$, $0 < s' < 1$, be a unit. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ be points such that $c = \alpha_0 < \alpha_1 < \cdots < \alpha_k = d$, and the discontinuity points of the function f are among them. We denote by Ω_i the intervals $\Omega_i := [\alpha_{i-1}, \alpha_i)$, $i = 1, \ldots, k$. The points α_i, $i = 0, \ldots, k-1$, are called the splitting points. We define a mapping $\varphi : \Omega \to \mathcal{A}$, where $\mathcal{A}$ is an alphabet $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$, by*

$$\varphi(x) = a_i \iff x \in \Omega_i.$$

Further, we define a mapping $w: s'\Omega \to \mathcal{A}^*$, which to any $x \in s'\Omega_i$ associates the word

$$w(x) = \varphi(x)\varphi(f(x))\varphi(f^{(2)}(x)) \ldots \varphi(f^{(n(x)-1)}(x)).$$

We say that a splitting $\Omega_1, \ldots, \Omega_k$, corresponding to splitting points $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$, is proper, if the mapping w is constant on each $s'\Omega_i$. The word $w(x)$, common for all $x \in s'\Omega_i$ is denoted by w_i. A proper splitting defines naturally a substitution rule $\zeta : \mathcal{A} \to \mathcal{A}^*$ by $\zeta(a_i) = w_i$.

Let us make several remarks on the above definition.

Remark 4.4.

(1) A splitting $\Omega_1, \ldots, \Omega_k$ is proper, if and only if for all $1 \leqslant i \leqslant k$, the following is true: for any $x \in s'\Omega_i$ there is a common value of $n(x) =: n_i$, and there exist indices $\ell_0, \ell_1, \ldots, \ell_{n_i-1}$ such that

$$s'\Omega_i \subset \Omega_{\ell_0} \qquad f(s'\Omega_i) \subset \Omega_{\ell_1} \qquad f^{(2)}(s'\Omega_i) \subset \Omega_{\ell_2}, \ldots, f^{(n_i-1)}(s'\Omega_i) \subset \Omega_{\ell_{n_i-1}}.$$

(2) For any $1 \leqslant i \leqslant k$, the interval Ω_i is a subset of either Ω_S, or Ω_M, or Ω_L. Thus the function f is continuous on each of the intervals Ω_i. It means that $s'\Omega_i \subset \Omega_{\ell_0}$ being an interval implies that $f(s'\Omega_i) \subset \Omega_{\ell_1}$ is an interval. Similarly, images $f^{(j)}(s'\Omega_i)$ of $s'\Omega_i$ under all iterations of f (up to $f^{(n_i-1)}$) are intervals.

(3) Note that if 0 belongs to Ω_{j_0} for some j_0, then $\varphi(0) = a_{j_0}$ and therefore the word $w(0)$ starts with a_{j_0}. Since 0 belongs also to the interval $s'\Omega_{j_0}$, and w is constant on $s'\Omega_{j_0}$, one has $w(0) = w_{j_0} = \zeta(a_{j_0})$. Similarly, let $z = f^{(-1)}(0)$, i.e. z' is the predecessor of 0 in $\Sigma_\beta(\Omega)$. We denote by i_0 the index for which $z \in \Omega_{i_0}$. Since $f^{(n_{i_0}-1)}(s'z) = z$, the word $w(z) = w_{i_0} = \zeta(a_{i_0})$ ends with a_{i_0}.

Definition 4.5. *We say that a substitution θ with the alphabet $\mathcal{B} = \{b_1, \ldots, b_k\}$ generates the quasicrystal $\Sigma_\beta(\Omega)$ from 0, if there exist indices $i_0, j_0 \in \{1, \ldots, k\}$, such that*

$$\theta^\infty(b_{i_0})|\theta^\infty(b_{j_0})$$

is a fixed point of the substitution θ, and it is possible to assign lengths to letters $b_1, \ldots, b_k$ in such a way that the Delone set corresponding to the bidirectional infinite word $\theta^\infty(b_{i_0})|\theta^\infty(b_{j_0})$, with the delimiter $|$ fixed at the origin, is the quasicrystal $\Sigma_\beta(\Omega)$.

Remark 4.6. Suppose that we have a proper splitting of the acceptance window Ω. This splitting assigns the letters, say a_{j_0}, a_{i_0}, to the origin and its predecessor in the quasicrystal. According to point 3 of remark 4.4, for the substitution ζ given by a proper splitting, we have $\zeta(a_{j_0}) = a_{j_0}w$, for some $w \in \mathcal{A}^*$, and $\zeta(a_{i_0}) = \tilde{w}a_{i_0}$, for some $\tilde{w} \in \mathcal{A}^*$. Therefore $\zeta^\infty(a_{i_0})|\zeta^\infty(a_{j_0})$ is a fixed point of the substitution ζ. If we assign the length X to letters a_i, such that $\Omega_i \subset \Omega_X$, for $X = S, M, L$, the Delone set which arises from the fixed point is the quasicrystal $\Sigma_\beta(\Omega)$.

The entire quasicrystal $\Sigma_\beta(\Omega)$ may arise by splitting the tiles in $s\Sigma_\beta(\Omega)$ correctly (according to the function f) by points of $\Sigma_\beta(\Omega)$. It means that except for the points of $s\Sigma_\beta(\Omega)$, we take also a suitable number of their successors in $\Sigma_\beta(\Omega)$. A point y of $s\Sigma_\beta(\Omega)$ corresponds to $x \in s'\Omega \cap \mathbb{Z}[\beta]$, where $x = y' = f^{(0)}(x)$. The successors of y correspond to points $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n(x)-1)}(x)$ in the acceptance window. We recall that the function $n(x)$ is constant on the intervals $s'\Omega_i$ of a proper splitting, i.e. we can denote $n(x) = n_i$, for any $x \in s'\Omega_i$. Therefore we have the disjoint union

$$\bigcup_{i=1}^{k}\bigcup_{j=0}^{n_i-1} f^{(j)}(s'\Omega_i \cap \mathbb{Z}[\beta]) = \Omega \cap \mathbb{Z}[\beta].$$

From the right continuity of the function f, we may also write

$$\bigcup_{i=1}^{k}\bigcup_{j=0}^{n_i-1} f^{(j)}(s'\Omega_i) = \Omega$$

where again the union is disjoint. According to point 1 of remark 4.4, all splitting points are boundary points of some of the intervals $f^{(j)}(s'\Omega_i)$. Now let a splitting point α_r be a boundary of an interval $f^{(j)}(s'\Omega_i) = [\alpha_r, \xi)$, for some $1 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant n_i - 1$. Then $s'[\alpha_{i-1}, \alpha_i) = f^{(-j)}[\alpha_r, \xi)$, and hence $\alpha_{i-1} = (s')^{-1}f^{(-j)}(\alpha_r)$. Note that j must be equal to p from proposition 4.2 and we have

$$\frac{1}{s'}f^{(-p)}(\alpha_r) = g(\alpha_r) = \alpha_{i-1} \tag{8}$$

where g was defined by (7). The following properties of g will be important.

Remark 4.7.

(1) For $x \in \mathbb{Z}[\beta]$, $g(x)$ is also an element of $\mathbb{Z}[\beta]$, i.e. if $x' \in \Sigma_\beta(\Omega)$, we have $(g(x))' \in \Sigma_\beta(\Omega)$.
(2) If $x \in s'\Omega$, then $g(x) = (s')^{-1}x$.
(3) For $c \notin s'\Omega$, i.e. $c \neq 0$, we have $g(f^{(-1)}(c)) = g(c)$. If $c = 0$, then $g(c) = c = 0$. $f^{(-1)}(0)$ is a predecessor of 0 in $\Sigma_\beta(\Omega)$ and thus $g(f^{(-1)}(0)) = f^{(-1)}(0)$. It is always true that 0 and the Galois image of its predecessor in $\Sigma_\beta(\Omega)$ are fixed points of g.
(4) Note that the equation (8) says that the set of splitting points is invariant with respect to g.

Proposition 4.8. *Let the set Γ of splitting points of Ω be finite. Then the splitting is proper, if and only if Γ is invariant with respect to g.*

Proof. The implication $\Rightarrow$ is given by 4 of remark 4.7. For the other implication ($\Leftarrow$) we recall that splitting points are always boundary points of intervals $f^{(j)}(s'\Omega_i)$, and therefore the entire interval $f^{(j)}(s'\Omega_i)$ is situated between splitting points. This implies that $f^{(j)}(s'\Omega_i)$ belongs to some Ω_ℓ. According to point 1 of remark 4.4, this has to say that the splitting is proper. $\square$

Proposition 4.9. *Let $\Omega = [c, d)$, $c, d \in \mathbb{Q}[\beta]$. We denote by γ_0, δ_0 the discontinuity points of the function f, and by*

$$\gamma_j := g^{(j)}(\gamma_0) \qquad j \in \mathbb{N}_0 \qquad \textit{and} \qquad \delta_j := g^{(j)}(\delta_0) \qquad j \in \mathbb{N}_0. \tag{9}$$

Then the set

$$\Gamma := \{c\} \cup \{\gamma_j \mid j \in \mathbb{N}_0\} \cup \{\delta_j \mid j \in \mathbb{N}_0\}$$

is finite, in particular, Γ is a proper splitting of Ω.

Note that since c, d belong to $\mathbb{Q}[\beta]$, so do the discontinuity points γ_0, δ_0 of the function f. Note also that c, γ_0 and δ_0 have to be splitting points according to the definition. Due to point 3 of remark 4.7, Γ is the minimal set containing c, γ_0 and δ_0, which is invariant with respect to the function g.

We recall that for certain lengths $d - c$ of the acceptance window the discontinuity points coincide and thus we have $\Gamma = \{c\} \cup \{\gamma_j \mid j \in \mathbb{N}_0\}$.

Proof. We first prove that $\{\gamma_j \mid j \in \mathbb{N}_0\}$ is finite, similar arguments then can be used to show that also $\{\delta_j \mid j \in \mathbb{N}_0\}$ is finite. Note that $\gamma_j \in [c, d)$ for all $j \in \mathbb{N}_0$.

From the relation (6), it follows that for any $x \in \mathbb{Q}[\beta] \cap [c, d)$ one has

$$x' - rL \leqslant (f^{(-r)}(x))' \leqslant x' \qquad \text{for} \quad r \geqslant 0.$$

Using the definition of functions f and g and the inequality above, one has

$$\gamma'_{j-1} - (K-1)L \leqslant s(g(\gamma_{j-1}))' = s\gamma'_j = (f^{(-p)}(\gamma_{j-1}))' \leqslant \gamma'_{j-1}.$$

(Note that the index p comes from the definition of g in (7) and is always less than or equal to $K - 1$.) By induction it is elementary to show the following inequality for $j = 1, 2, \ldots$:

$$-|\gamma'_0| - \frac{s(K-1)L}{s-1} \leqslant \frac{1}{s^j}\gamma'_0 - \frac{s(K-1)L}{s-1} \leqslant \gamma'_j \leqslant \frac{1}{s^j}\gamma'_0 \leqslant |\gamma'_0|. \tag{10}$$

Now $\gamma_0 \in \mathbb{Q}[\beta]$, thus there exists $q \in \mathbb{N}$, such that $\gamma_0 \in \frac{1}{q}\mathbb{Z}[\beta]$, and according to properties of f and g, also $\gamma_j \in \frac{1}{q}\mathbb{Z}[\beta]$. Since $\gamma_j \in [c, d)$, we have $q\gamma_j \in [qc, qd)$. Thus $q\gamma_j \in [qc, qd) \cap \mathbb{Z}[\beta]$, i.e. $(q\gamma_j)' = q\gamma'_j$ belongs to the quasicrystal $\Sigma_\beta[qc, qd)$. Such a set is Delone and therefore it has only finitely many points in a bounded interval. It follows from (10) that

$$q\left(-|\gamma'_0| - \frac{s(K-1)L}{s-1}\right) \leqslant q\gamma'_j \leqslant q|\gamma'_0|.$$

Therefore $q\gamma'_j$ belongs to a bounded interval for any j. Thus there are only finitely many values in the set $\{\gamma_j \mid j \in \mathbb{N}_0\}$. The argument is identical for finiteness of the set $\{\delta_j \mid j \in \mathbb{N}_0\}$.

We have shown that Γ is finite. Due to point 3 of remark 4.7, it is invariant with respect to g. Thus we may use proposition 4.8 to conclude that Γ defines a proper splitting of Ω. $\square$

The previous proposition states that under certain conditions on the acceptance interval $\Omega = [c, d)$ there are only finitely many splitting points. Their number determines the cardinality of the substitution alphabet. The main result of the paper is formulated as the following theorem.

Theorem 4.10. *Let β be a quadratic Pisot unit. There exists an alphabet $\mathcal{A} = \{a_1, \ldots, a_k\}$, $k \geqslant 2$, and a substitution $\theta : \mathcal{A}^* \to \mathcal{A}^*$, generating the quasicrystal $\Sigma_\beta[c, d)$ starting from 0 if and only if $c, d \in \mathbb{Q}[\beta]$.*

Proof. If $c, d \in \mathbb{Q}[\beta]$, the substitution exists. We construct the set Γ, which is a proper splitting of Ω, according to propositions 4.8 and 4.9. It was shown in remark 4.6 that a proper splitting of the acceptance window defines a substitution with a finite alphabet, which generates the quasicrystal $\Sigma_\beta(\Omega)$ starting from the origin.

On the other hand, suppose that there exists a substitution rule θ with an alphabet $\mathcal{A} = \{a_1, \ldots, a_k\}$, generating the quasicrystal $\Sigma_\beta(\Omega)$. Necessarily, the Perron–Frobenius eigenvalue s of the substitution (scaling factor of the quasicrystal) belongs to $\mathbb{Z}[\beta]$, and $s > 1$. Without loss of generality $s' > 0$, otherwise we consider the second iterate of θ, which has the eigenvalue s^2. Since $s\Sigma_\beta(\Omega) \subset \Sigma_\beta(\Omega)$, we have $s'\Omega \subset \Omega$, and thus $|s'| < 1$. Note that $s\Sigma_\beta(\Omega) = \Sigma_\beta(s'\Omega) \cap (s\mathbb{Z}[\beta])$.

The substitution determines a splitting of Ω into k intervals $\Omega_1, \ldots, \Omega_k$ according to letters $a_1, \ldots, a_k$ of the alphabet. Let us denote the splitting points by $c = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k = d$. Similarly, as in example 3.1, the substitution corresponds to the action of a finite number of affine mappings

$$t_{(j)}x := sx + h_j$$

where s is the scaling factor and h_j is a translation in $\mathbb{Z}[\beta]$. Corresponding action in the acceptance window is carried out by mappings

$$t^*_{(j)}x := s'x + h'_j.$$

For the substitution of a particular letter a_i we use only certain of the mappings $t_{(j)}$. For those $t_{(j)}$, there exists an index n, such that

$$t^*_{(j)}(s'(\Omega_i \cap \mathbb{Z}[\beta])) \subset \Omega_n. \tag{11}$$

Since the substitution θ generates from $s\Sigma_\beta(\Omega)$ the entire quasicrystal $\Sigma_\beta(\Omega)$, necessarily

$$\Omega \cap \mathbb{Z}[\beta] = \bigcup_{j,i} t^*_{(j)}(s'(\Omega_i \cap \mathbb{Z}[\beta])) \tag{12}$$

where this is a disjoint union over suitable indices i, j. From (11) and (12) we find that for any splitting point α_j, $j = 0, 1, \ldots, k$, there exists an α_i, $i = 0, 1, \ldots, k$, and an index $r_{i,j}$, such that

$$\alpha_j = s'\alpha_i + h'_{r_{i,j}}.$$

The latter is a system of linear equations with parameters $h'_{r_{i,j}}$ for the variables α_i, $i = 0, \ldots, k$. The matrix of the system is diagonally dominant and has entries in $\mathbb{Z}[\beta]$. Such a system of linear equations has a unique solution $(\alpha_0, \ldots, \alpha_k)$ given by the Cramer's rule, i.e. each α_i equals the ratio of two determinants with values in $\mathbb{Z}[\beta]$. Therefore $\alpha_i \in \mathbb{Q}[\beta]$ for $i = 0, \ldots, k$. In particular, both $c = \alpha_0$, and $d = \alpha_k$ belong to $\mathbb{Q}[\beta]$. □

Finding a substitution generating $\Sigma_\beta(\Omega)$ from its point x is equivalent to finding a rule generating $\Sigma_\beta(\Omega - x')$ from the origin. The existence of such a substitution rule implies, according to the above theorem, that $\Omega - x'$ has boundaries in $\mathbb{Q}[\beta]$. Obviously, the set $\Omega = (\Omega - x') + x'$ also has boundaries in $\mathbb{Q}[\beta]$. Consequently, we have the following corollary.

Corollary 4.11. *If there exists a substitution generating the quasicrystal $\Sigma_\beta(\Omega)$ from a point $x \in \Sigma_\beta(\Omega)$, then for any $y \in \Sigma_\beta(\Omega)$, there exists a substitution generating $\Sigma_\beta(\Omega)$ from y.*

Generally the alphabet of the substitution is large, depending on the position of the quasicrystal acceptance window $[c, d)$. A number of examples with $c, d \in \mathbb{Z}[\beta]$ show that the cardinality of the alphabet depends on the length of β-expansions of c, d. For the details of β-expansions of numbers see [12].

In certain cases it is possible to reduce the alphabet of a given substitution. Let us illustrate this possibility on the substitution of the example 3.2. Consider the quasicrystal $\Sigma[0, 1+1/\tau^2)$ in $\mathbb{Z}[\tau]$. Its substitution ζ may be found in figure 2. The second iteration of ζ is

$$\begin{aligned} S &\mapsto SM_2LM_1 \\ L &\mapsto SM_2LM_1SM_2LM_1SM_2M_1 \\ M_1 &\mapsto SM_2LM_1SM_2M_1 \\ M_2 &\mapsto SM_2LM_1SM_2M_1. \end{aligned}$$

It is obvious that the letters M_1 and M_2 are substituted by the same sequence of letters and therefore one may identify them. We obtain the substitution with the alphabet $\{S, L, M\}$

$$\begin{aligned} S &\mapsto SMLM \\ L &\mapsto SMLMSMLMSMM \\ M &\mapsto SMLMSMM. \end{aligned}$$

It would be interesting to determine in which cases such a reduction of the alphabet of the substitution is possible, namely, to answer the following question: for which acceptance windows Ω does there exist a three-letter substitution generating the corresponding quasicrystal $\Sigma_\beta(\Omega)$?

5. Types of acceptance intervals

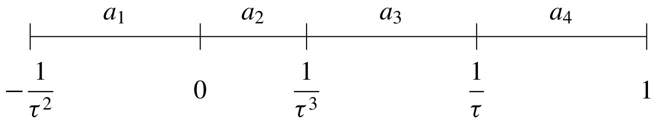
In previous sections we have explained how to construct substitution rules for quasicrystals with acceptance windows $[c, d)$, $c, d \in \mathbb{Q}[\beta]$. Let us examine how the presence or absence of boundary points in the acceptance window may influence the substitution rule. If the boundary point $d \in \mathbb{Q}[\beta]$ does not belong to the ring $\mathbb{Z}[\beta]$, then $\Sigma_\beta[c, d) = \Sigma_\beta[c, d]$. Similarly, if $c \in \mathbb{Q}[\beta]$, $c \notin \mathbb{Z}[\beta]$, we have $\Sigma_\beta[c, d) = \Sigma_\beta(c, d)$. Thus if at least one of the boundary points of the acceptance interval is not an element of $\mathbb{Z}[\beta]$, it is always possible to use the presented method, considering the acceptance window $\Omega = [c, d)$, or $\Omega = (c, d]$. In both cases, the function f from (5) is continuous at least from one side, right continuous or left continuous, respectively.

If c, d are both elements of the ring $\mathbb{Z}[\beta]$, and the given quasicrystal has an open or closed acceptance window ($\Omega = (c, d)$ or $\Omega = [c, d]$), we need a special approach. The discontinuity points of the function f must be assigned by additional letters, and the resulting substitution rule is not primitive. The letters of the alphabet are divided into two groups, $\{a_1, \dots, a_k\}$, corresponding to interiors of the intervals Ω_i of the splitting, and $\{b_1, \dots, b_r\}$, corresponding to splitting points. Note that $r = k + 1$ for $\Omega = [c, d]$, and $r = k - 1$ for $\Omega = (c, d)$. Clearly, by the same procedure as for the case of semiclosed intervals, we assign to all letters a_i words formed only by letters a_j, $j = 1, \dots, k$. Together with words assigned to letters b_i, we have a substitution rule with the alphabet $\{a_1, \dots, a_k\} \cup \{b_1, \dots, b_r\}$. Its matrix has the block form

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}.$$

Since any power P^n has the same form as P, the substitution is not primitive. In particular it means that some of the letters appear only finitely many times in a fixed point of the substitution.

Example 5.1. In here, we illustrate the difference between substitution rules for quasicrystals $\Sigma_\tau(\Omega)$ with acceptance windows $[-1/\tau^2, 1)$, $(-1/\tau^2, 1]$, $(-1/\tau^2, 1)$, $[-1/\tau^2, 1]$. Using the algorithm described in sections 3 and 4 for the quasicrystal $\Sigma[-1/\tau^2, 1)$, we obtain the following splitting of the acceptance window:



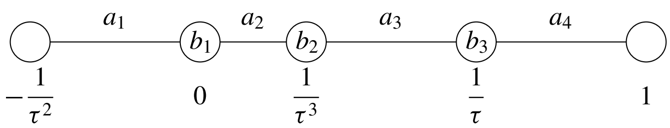
Quasicrystal points x, such that $x' \in [-1/\tau^2, 0)$, are assigned with the letter a_1; similarly, $[0, 1/\tau^3) \to a_2$, $[1/\tau^3, 1/\tau) \to a_3$, and $[1/\tau, 1) \to a_4$, as illustrated in the splitting diagram above. Using this assignment of letters, we have the substitution rule ζ from definition 4.3:

$$\begin{aligned} a_1 &\mapsto a_1a_4 \\ a_2 &\mapsto a_2a_3a_1a_4 \\ a_3 &\mapsto a_2a_3 \\ a_4 &\mapsto a_3a_1a_4. \end{aligned} \tag{13}$$

Using the lengths 1 for a_1, τ^2 for a_2 and τ for a_3, a_4, the quasicrystal $\Sigma[-1/\tau^2, 1)$ corresponds to the fixed point $\zeta^\infty(a_4)|\zeta^\infty(a_2)$. The starting points were chosen according to point 3 of remark 4.4.

For the acceptance window $(-1/\tau^2, 1]$, the splitting is the same as before. However, the assignment of letters is different: $(-1/\tau^2, 0] \to a_1$, $(0, 1/\tau^3] \to a_2$, $(1/\tau^3, 1/\tau] \to a_3$, and $(1/\tau, 1] \to a_4$. The substitution rule coincides with that of (13), but the starting points have to be chosen differently. The quasicrystal $\Sigma(-1/\tau^2, 1]$ is the fixed point $\zeta^\infty(a_3)|\zeta^\infty(a_1)$.

Consider now the open interval $(-1/\tau^2, 1)$. The letters $a_1, \ldots, a_4$ are assigned to the open intervals. For their boundaries, we have to introduce three additional letters b_1, b_2, b_3, $0 \to b_1$, $1/\tau^3 \to b_2$, and $1/\tau \to b_3$:

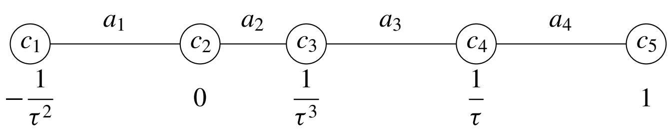


The substitution rule (13) has to be extended by

$$\begin{aligned} b_1 &\mapsto b_1a_3a_1a_4 \\ b_2 &\mapsto a_2a_3a_1a_4 \\ b_3 &\mapsto b_2b_3 \end{aligned} \tag{14}$$

where the words were obtained in a standard way using the function f of (5). If we assign the lengths τ^2 to b_1, b_2, and τ to b_3, the quasicrystal $\Sigma(-1/\tau^2, 1)$ is the fixed point $\zeta^\infty(b_3)|\zeta^\infty(b_1)$ of the substitution ζ given by (13) and (14), with the alphabet $\{a_1, \ldots, a_4, b_1, b_2, b_3\}$.

Finally, let us consider the quasicrystal $\Sigma[-1/\tau^2, 1]$. The boundary points of the intervals are assigned by letters $-1/\tau^2 \to c_1$, $0 \to c_2$, $1/\tau^3 \to c_3$, $1/\tau \to c_4$, and $1 \to c_5$:



Using the function f we obtain the substitution rule

$$\begin{aligned} c_1 &\mapsto a_1a_4 \\ c_2 &\mapsto c_2c_5 \\ c_3 &\mapsto a_2a_3 \\ c_4 &\mapsto c_3c_1c_4 \\ c_5 &\mapsto a_3a_1a_4 \end{aligned} \tag{15}$$

which together with (13) gives the fixed point $\zeta^\infty(c_4)|\zeta^\infty(c_2)$. If the length of c_1, c_2 is 1, and the length of c_3, c_4, c_5 is τ, the latter fixed point coincides with the quasicrystal $\Sigma[-1/\tau^2, 1]$.

If all the splitting points have their own letters, we obtain a substitution rule which generates the given quasicrystal. However, in the majority of cases, we may reduce the number of elements in the alphabet, if we examine carefully whether, while iterating the function f, the boundary point does not behave in the same way as an interval adjacent to it. For example, in the rule given by (13) and (15), generating the quasicrystal $\Sigma[-1/\tau^2, 1]$, one may identify $c_1 \equiv a_1$, $c_3 \equiv a_3$, and $c_5 \equiv a_4$. Then, it suffices to add to (13) the rules $c_2 \mapsto c_2a_4$, $c_4 \mapsto a_3a_1c_4$.

We recall that the substitution process starts with the pair $c_4|c_2$. From the reduced prescription, it is clear that the letters c_4, c_2 occur only once in the entire infinite word, namely around the origin, denoted by the delimiter |. It means that the densities of letters c_2, c_4 in the infinite word vanish. This can also be justified using the substitution matrix of the rule

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

First, note that the matrix P is decomposable. It means that any of its powers contain zero elements. Such a matrix is not primitive. Both P and P^T have the eigenvalue τ^2 (scaling factor of the quasicrystal). The corresponding eigenvector of P^T has zero components in positions c_2, c_4. These are the densities of c_2, c_4 in the fixed point.

6. Summary and concluding remarks

In this paper we have shown that for every quadratic Pisot unit β there is an infinite family of non-equivalent quasicrystals in $\mathbb{Z}[\beta]$, which may be generated by substitution rules. The substitutions generically have a reducible characteristic polynomial.

Using the results of this paper, let us now underline three steps which allow one to construct the substitution rules for a given quasicrystal $\Sigma_\beta(\Omega)$ and a given seed point $x \in \Sigma_\beta(\Omega)$. First recall that the boundary points c, d of the $\Omega = [c, d)$ must be in $\mathbb{Q}[\beta]$. Then we proceed as follows.

(i) First we use the function g given by (7) to generate the set Γ of splitting points according to proposition 4.9.

(ii) The splitting Γ is proper and thus provides a substitution ζ as in definition 4.3.

(iii) In order to decide, which letters should be used to start the generation on the right and on the left, we use item 3 of remark 4.4.

Example 6.1. In our final example we derive the substitution rules for the quasicrystal $\Sigma_\beta(\Omega)$ with $\beta = \frac{1}{2}(5 + \sqrt{21})$, solution of $x^2 = 5x - 1$. For the acceptance window we choose the interval $\Omega = [2\beta - 10, 1)]$. We find the substitution rule generating $\Sigma_\beta(\Omega)$ starting from the seed point $x = 0$.

First we have to find the distances between adjacent points in the quasicrystal $\Sigma_\beta(\Omega)$, depending on the length $|\Omega| = 11 - 2\beta$ of the acceptance interval. Using proposition 2.5 we find that the length of tiles in the chosen quasicrystal is 1, β and $\beta - 1$. The next step is to find the expression for the stepping function in this quasicrystal. It is given by

$$f(x) = \begin{cases} x+1 & \text{for} \quad x \in [2\beta - 10, 0) \\ x+5-\beta & \text{for} \quad x \in [0, 3\beta - 14) \\ x+4-\beta & \text{for} \quad x \in [3\beta - 14, 1). \end{cases}$$

Note that the points of discontinuity of the function f are 0 and $3\beta - 14$. They serve as initial points for deriving the set Γ of splitting points. For the scaling factor we may consider $s = \beta$, since $s' = \beta' > 0$ is a self-similarity factor of the quasicrystal $\Sigma_\beta(\Omega)$. We find

$$\Gamma = \{2\beta - 10, 0, 3\beta - 14, \beta - 4\}.$$

This splitting is proper in the sense of definition 4.3, so that we can find the substitution ζ given by the proper splitting Γ, as

$$\begin{aligned} S &\mapsto SM_2 \\ L &\mapsto LLM_1SM_1SM_2 \\ M_1 &\mapsto LLM_1SM_1 \\ M_2 &\mapsto LM_1SM_1SM_2. \end{aligned}$$

As the starting pair for generation of the quasicrystal $\Sigma_\beta(\Omega)$ from the origin we have to take $M_2|S$, according to point 3 of remark 4.4.

The result of this paper is a method of symbolically generating three-tile quasicrystals for all the family of quadratic Pisot irrationalities. Our motivation in this study was the need to generate a large number of quasicrystal points with absolute precision. Generally methods used for generation of quasiperiodic structures can be split into two classes according to the way its elements, points or tiles are constructed. The first method is numerical, the second is symbolic. As long as a small number of such elements need to be found, either way can serve the purpose. The situation is quite different, even in one dimension, when the generation of elements has to proceed for a long time and a large number of points have to be found. Then each of the methods encounters very different practical difficulties. In the first case the difficulty is the well known task of deciding which of two numbers is greater when their values are extremely close to each other. In the second case, the construction is symbolic, hence it proceeds with absolute precision. However, in extensive computations, there is evidently a problem with an ever-increasing need to remember larger and larger structures which have been generated. An efficient algorithm for generating fixed points of substitutions can be designed [13]. The memory requirements of such a generator increase only logarithmically with the number of generated elements of the infinite word.

In this paper we have studied substitutions with quadratic Perron–Frobenius eigenvalues. The problems invoked by quasicrystals with higher than quadratic irrationalities are much more complicated. The definition of one-dimensional quasicrystals based on an irrationality of degree n requires an $(n - 1)$-dimensional acceptance window. An example of such a quasicrystal is the set generated by a substitution acting on n symbols whose characteristic polynomial is the minimal polynomial of the considered irrationality. As shown in [7], the boundary of the acceptance window of such a point set is generically an anisotropic fractal.

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